Int. J. Solids Structures Vol. 24, No. 10, pp. 1003–1019, 1988
Printed in Great Britain.

0020-7683/88 $3.00+.00


# LATERAL BUCKLING OF *I* BEAMS WITH WEB STIFFENERS AND BATTEN PLATES

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(*Received* 30 *October* 1987; *in revised form* 16 *April* 1988)

**Abstract**—This paper presents a general discussion of the lateral buckling of uniform and doubly symmetric *I* beams with web stiffeners and batten plates. In order to express generally the lateral buckling strength of the *I* beam, whether with or without the web stiffeners and/or batten plates, a function extending the Dirac function is used. The critical moments derived from the energy method take the same equations as ones for the original *I* beam when the torsional constant and the moment of inertia about the weak axis of the *I* beam with the web stiffeners and or batten plates are exchanged for those of the original *I* beam. The numerical results show that, in practice, the effects due to the web stiffeners and batten plates on the moment of inertia about the weak axis can be neglected, but the effects due to those on the torsional constant cannot be neglected. This paper demonstrates mathematically the phenomenon that the web stiffeners and batten plates increase the critical moment of beams.

## 1. INTRODUCTION

Lateral buckling of uniform beams without web stiffeners and batten plates has been studied by, for example, Vlasov (1936), Timoshenko and Gere (1961), Galambos (1936), Salvadori (1956) and Bleich (1952). However, in practice, beams are accompanied by web stiffeners and/or batten plates, and it is generally known that web stiffeners and batten plates increase the lateral buckling strength of beams. This increase is considered to be due to the local increment of both the torsional stiffness and the bending stiffness resulting from the use of web stiffeners and/or batten plates. Nevertheless, the present trends in structural design systems neglect the effects due to web stiffeners and batten plates on lateral buckling. Also, the relationship between beams with and without web stiffeners and/or batten plates in terms of lateral buckling is not clear.

Although Vlasov (1936) discusses a design method for beams with web stiffeners and/or batten plates, there is no mention of lateral buckling, and the design method is not practical because of the complicated equations. Thus, although there are a lot of papers dealing with the lateral buckling of uniform *I* beams, there are few studies dealing with the lateral buckling of the *I* beams with web stiffeners and/or batten plates.

The purpose of this paper is to give a general view of the lateral buckling of a uniform and symmetric *I* beam with web stiffeners and/or batten plates. In order to obtain a general description of the lateral buckling of a beam, whether with or without the web stiffeners and the batten plates, the local effects due to web stiffeners and batten plates are expressed by using a function extending the Dirac function. The lateral buckling of *I* beams with web stiffeners and batten plates is presented by using the energy method; and the obtained equation will be similar to the well-known equation of *I* beams without web stiffeners and batten plates when the torsional constant and the moment of inertia about the weak axis of the beam are exchanged for those of the original *I* beam. Next, the expressions presented for the lateral buckling are verified by numerical calculations.

## 2. BASIC ASSUMPTIONS

In analyses the following assumptions are used.

(1) The beam is a doubly symmetric and uniform *I* beam.

(2) The in-plane distortion of the transverse cross section, the initial imperfection, and the residual stresses are neglected.

(3) The lateral buckling is in an elastic region.

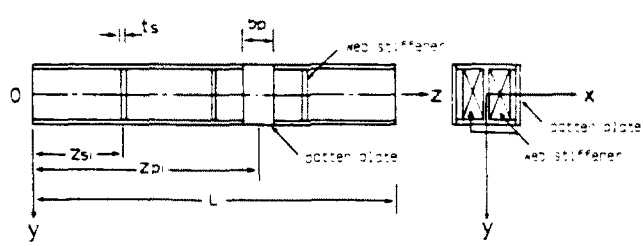


Fig. 1. Coordinates of a beam with web stiffeners and batten plates.

## 3. TORSIONAL CONSTANT AND MOMENT OF INERTIA OF A BEAM

Consider a doubly symmetric and uniform $I$ beam with web stiffeners and batten plates as shown in Fig. 1, in which the members and location of the web stiffeners and batten plates can be taken arbitrarily. The Cartesian coordinate system is employed, in which the axes $x$ and $y$ are the principal axes of the beam, and the axis $z$ passes through the centroidal axis of the beam.

The torsional constant, $J$, and moment of inertia about its weak axis, $I_y$, at each point where the web stiffener or batten plate is located increase locally due to the web stiffener or batten plate. The moment of inertia about its weak axis, $I_y$, and the torsional constant, $J$, of the beam in which a web stiffener is located at $z = z_{si}$ and a batten plate at $z = z_{pi}$, may be expressed generally as

$$I_y = I_{y0} + \sum_{i=1}^{s} I_{ys}\, D(z - z_{si}) + \sum_{i=1}^{p} I_{yp}\, D(z - z_{pi}) \tag{1}$$

$$J = J_0 + \sum_{i=1}^{s} J_s\, D(z - z_{si}) + \sum_{i=1}^{p} J_p\, D(z - z_{pi}) \tag{2}$$

in which $I_{y0}$ is the moment of inertia about the weak axis of the $I$ beam without the web stiffeners and batten plates, $I_{ys}$ the moment of inertia about the weak axis due to a web stiffener located at $z = z_{si}$; $I_{yp}$ the moment of inertia about the weak axis due to a batten plate located at $z = z_{pi}$; $J_0$ the torsional constant of the $I$ beam without web stiffeners and batten plates; $J_s$ the torsional constant due to a web stiffener located at $z = z_{si}$; $J_p$ the torsional constant due to a batten plate located at $z = z_{pi}$; $z_{si}$ the distance of the middle point of thickness of the $i$th web stiffener; $z_{pi}$ the distance of the middle point of width of the $i$th batten plate; $\sum^{s}$ the sum for the total number, $s$, of web stiffeners; and $\sum^{p}$ the sum for the total number, $p$, of batten plates. Also, $D(z - z_{si})$ is defined as a function in which the Dirac function $\delta(z - z_{si})$ exists continuously in the region of thickness, $t_s$, of the $i$th web stiffener, namely, the region from $z_{si} - t_s/2$ to $z_{si} + t_s/2$. Similarly, $D(z - z_{pi})$ is defined as a function in which the Dirac function $\delta(z - z_{pi})$ exists continuously in the region of width, $b_p$, of the $i$th batten plate, namely, the region from $z_{pi} - b_p/2$ to $z_{pi} + b_p/2$. Briefly, function $D$ is considered to be the sum of the Dirac function $\delta$ distributing continuously in the given region. The detail and employment of the Dirac function are given in Mikusinski and Sikorski (1957) and Frýba (1972), respectively.

In order to state briefly the subsequent development, it is assumed that each shape of the web stiffeners and batten plates at an arbitrary point of the $I$ beam is symmetric with respect to the $y$-axis, as shown in Fig. 2. Hence, each web stiffener and each batten plate make a pair at points where they are located. The transverse cross section of the $I$ beam at the point where a pair of web stiffeners is located is translated from the $I$ section to the solid section due to the existence of their web stiffeners. Hence, the torsional constant, $J_s$, due to their web stiffeners can be obtained by subtracting the torsional constant of the original $I$ beam from the torsional constant of the solid section. On the other hand, the transverse cross section of the $I$ beam at the point where a pair of batten plates is located forms the closed cross section by the $I$ beam and their batten plates. Hence, the torsional

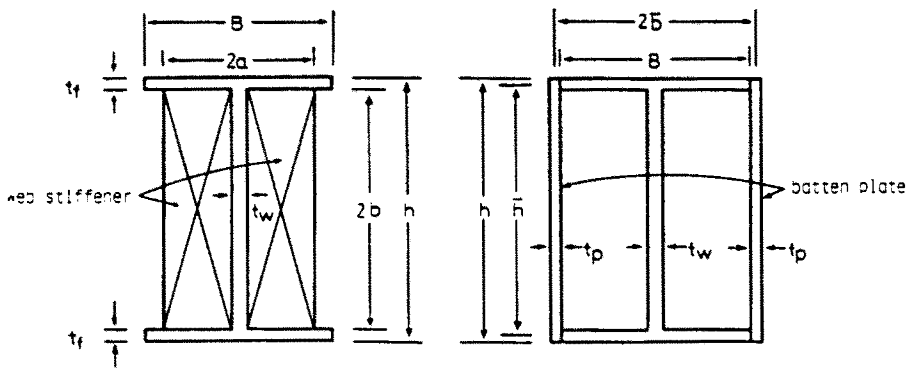


Fig. 2. Web stiffeners and batten plates.

constant, $J_p$, due to their batten plates, is obtained by subtracting the torsional constant of the original $I$ beam from the torsional constant of the closed cross section. Meanwhile, the increments of the moments of inertia due to a pair of web stiffeners and a pair of batten plates are easily obtained from a consideration of the increments due to their web stiffeners and batten plates, respectively. Hence, the torsional constants, $J_s$ and $J_p$, and the moments of inertia, $I_{ys}$ and $I_{yp}$, are given as

$$J_s = a^3 \frac{h}{2} f_1 - \frac{2b}{3}(t_w)^3 - \frac{4a}{3}(t_f)^3 \tag{3}$$

$$J_p = \frac{8(\bar{b}\bar{h})^2}{\dfrac{\bar{h}}{t_p} + \dfrac{2\bar{b}}{t_f}} + \frac{2}{3} h t_p^3 \tag{4}$$

$$I_{ys} = \frac{2b(2a)^3}{12} - \frac{2b t_w^3}{12} \tag{5}$$

$$I_{yp} = \frac{2h t_p^3}{12} + 2t_p h \bar{b}^2 \tag{6}$$

in which the constant $f_1$ takes the following value from Utoguch (1968):

$$f_1 = \frac{16}{3}\left[1 - \frac{192}{\pi^5 \left(\dfrac{h}{2a}\right)} \sum_{n=1}^{\infty} \frac{1}{n^5} \tanh \frac{n\pi h}{4a}\right] \quad (n = 1, 3, 5, \ldots, \infty) \tag{7}$$

in which constants $a$, $b$, $t_f$, $t_w$, $h$, $\bar{h}$, $\bar{b}$ are indicated in Fig. 2. Here, eqns (3) and (4) are easily obtained from Gjelsvik (1981).

Since the web stiffeners and batten plates are located discontinuously in the beam, they do not have enough stiffness in the axial direction. Therefore, the web stiffeners and the batten plates may be assumed to have no effect on warping in practice, and the warping constant, $\Gamma$, can be taken as the value of the original $I$ beam, namely

$$\Gamma = \frac{\bar{h}^2}{24} B^3 t_f. \tag{8}$$

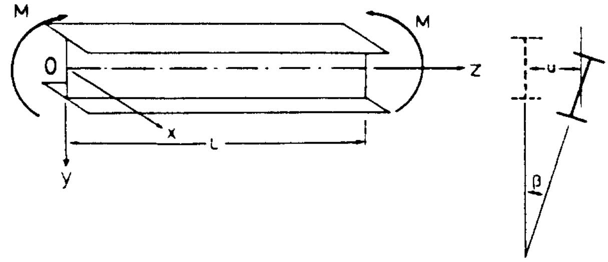


Fig. 3. Lateral buckling of $I$ beam in pure bending.

## 4. BUCKLING OF SIMPLY SUPPORTED $I$ BEAMS IN UNIFORM BENDING

Consider the lateral buckling of the $I$ beam, subject to a uniform bending moment, $M$, as shown in Fig. 3.

From eqn (5.87) of Chajes (1974), the total potential energy, $\Pi_B$, stored in the beam during buckling is given as

$$\Pi_B = \frac{1}{2}\int_0^L [E I_y (u'')^2 + E\Gamma(\beta'')^2 + G J(\beta')^2 - 2M u' \beta'] \, dz \tag{9}$$

in which $u$ is the displacement in the $x$ direction, $\beta$ the angle of twist about the $z$ axis, $E$ Young's modulus, $G$ the shear modulus, and $L$ the span length of the beam. Primes indicate differentiation with respect to $z$. Substituting eqns (1) and (2) into eqn (9) yields

$$\Pi_B = \frac{1}{2}\int_0^L \left\{ E\left[ I_{y0} + \sum_{i=1}^{s} I_{ys} D(z - z_{si}) + \sum_{i=1}^{p} I_{yp} D(z - z_{pi}) \right](u'')^2 \right.$$
$$\left. + E\Gamma(\beta'')^2 + G\left[ J_0 + \sum_{i=1}^{s} J_s D(z - z_{si}) + \sum_{i=1}^{p} J_p D(z - z_{pi}) \right](\beta')^2 - 2M u' \beta' \right\} dz. \tag{10}$$

The ends of the beam are assumed to be restrained against twisting and lateral translation, but free to warp and free to rotate about the principal axes. The analytical formulation of these boundary conditions is

$$\left. \begin{aligned} u = u'' &= 0 \\ \beta = \beta'' &= 0 \end{aligned} \right\} \quad \text{at } z = 0 \text{ and } L. \tag{11}$$

In order to present briefly the subsequent development, the buckling deformations $u$ and $\beta$ satisfying eqns (11) are approximated as

$$u = u_1 \sin\frac{\pi z}{L} \tag{12}$$

$$\beta = \beta_1 \sin\frac{\pi z}{L} \tag{13}$$

in common with Chajes (1974).

Substituting eqns (12) and (13) into eqn (10), the total energy, $\Pi_B$, yields

$$\Pi_B = \frac{1}{4}\frac{\pi^2}{L}\left[ E\left(\frac{\pi}{L}\right)^2 (u_1)^2 I_{y1} + E\Gamma\left(\frac{\pi}{L}\right)^2 (\beta_1)^2 + G(\beta_1)^2 J_1 - 2M u_1 \beta_1 \right] \tag{14}$$

in which $I_{y1}$ and $J_1$ are defined as

$$I_{y1} = I_{y0} + \frac{2}{L}\sum_{i=1}^{s}\int_0^L I_{ys}\, D(z - z_{si})\sin^2\left(\frac{\pi z}{L}\right)dz + \frac{2}{L}\sum_{i=1}^{p}\int_0^L I_{yp}\, D(z - z_{pi})\sin^2\left(\frac{\pi z}{L}\right)dz \tag{15}$$

$$J_1 = J_0 + \frac{2}{L}\sum_{i=1}^{s}\int_0^L J_s\, D(z - z_{si})\cos^2\left(\frac{\pi z}{L}\right)dz + \frac{2}{L}\sum_{i=1}^{p}\int_0^L J_p\, D(z - z_{pi})\cos^2\left(\frac{\pi z}{L}\right)dz. \tag{16}$$

Using the following formulas for the Dirac function $\delta$

$$\int_a^b \delta(z-\xi)\, f(z)\, dz = \begin{cases} 0 & \text{for } \xi < a < b \\ f(\xi) & \text{for } a < \xi < b \\ 0 & \text{for } a < b < \xi \end{cases} \tag{17}$$

the terms including function $D$ in eqns (15) and (16) become

$$\begin{aligned} \int_0^L J_s\, D(z - z_{si})\cos^2\left(\frac{\pi z}{L}\right)dz &= \int_{z_{si}-t_s/2}^{z_{si}+t_s/2}\left[\int_0^L J_s\,\delta(z-\bar{z})\cos^2\left(\frac{\pi z}{L}\right)dz\right]d\bar{z} \\ &= \int_{z_{si}-t_s/2}^{z_{si}+t_s/2} J_s(\bar{z})\cos^2\left(\frac{\pi \bar{z}}{L}\right)d\bar{z} \\ &= \frac{J_s}{2}t_s\left[1 + \frac{L}{\pi t_s}\sin\left(\frac{\pi t_s}{L}\right)\cos\left(\frac{2\pi z_{si}}{L}\right)\right]. \end{aligned} \tag{18}$$

In order to obtain the approximation of eqn (18), expanding $\sin(\pi t_s/L)$ in power series, eqn (18) can be rewritten as

$$\int_0^L J_s\, D(z - z_{si})\cos^2\left(\frac{\pi z}{L}\right)dz = \frac{J_s}{2}t_s\left\{1 + \left[1 - \frac{1}{3!}\left(\frac{\pi t_s}{L}\right)^2 + \frac{1}{5!}\left(\frac{\pi t_s}{L}\right)^4 + \cdots\right]\cos\left(\frac{2\pi z_{si}}{L}\right)\right\}. \tag{19}$$

Since this series converges very rapidly, the consideration of the first term of the series gives an accuracy sufficient for all practical purposes. Hence, the approximation of eqn (18) may be written as

$$\int_0^L J_s\, D(z - z_{si})\cos^2\left(\frac{\pi z}{L}\right)dz \simeq J_s t_s \cos^2\left(\frac{\pi z_{si}}{L}\right). \tag{20}$$

Similarly, one can obtain the following expression:

$$\int_0^L I_{ys}\, D(z - z_{si})\sin^2\left(\frac{\pi z}{L}\right)dz \simeq I_{ys} t_s \sin^2\left(\frac{\pi z_{si}}{L}\right). \tag{21}$$

From eqns (20) and (21), the function $D$ has the following approximate relations with the Dirac function $\delta$:

$$D(z-z_{si}) \simeq t_s\,\delta(z-z_{si}) \tag{22}$$

$$D(z-z_{pi}) \simeq b_p\,\delta(z-z_{pi}). \tag{23}$$

Using these relations, eqns (15) and (16) can be expressed as

$$I_{y1} = I_{y0} + \underline{\frac{2}{L}\sum_{i=1}^{s} I_{ys}\,t_s \sin^2\left(\frac{\pi z_{si}}{L}\right) + \frac{2}{L}\sum_{i=1}^{p} I_{yp}\,b_p \sin^2\left(\frac{\pi z_{pi}}{L}\right)} \tag{24}$$

$$J_1 = J_0 + \underline{\frac{2}{L}\sum_{i=1}^{s} J_s\,t_s \cos^2\left(\frac{\pi z_{si}}{L}\right) + \frac{2}{L}\sum_{i=1}^{p} J_p\,b_p \cos^2\left(\frac{\pi z_{pi}}{L}\right)} \tag{25}$$

in which the underlined terms indicate the effects due to the web stiffeners and batten plates. If the web stiffeners and batten plates have the same shapes, the moments of inertia, $I_{ys}$ and $I_{yp}$, and the torsional constants, $J_s$ and $J_p$, due to the web stiffeners and the batten plates, respectively, become constants.

The critical moment, $M$, is reached when neutral equilibrium is possible, and the requirement for neutral equilibrium is that the derivatives of $\Pi_B$ with respect to $u_1$ and $\beta_1$ vanish

$$\frac{\partial \Pi_B}{\partial u_1} = 0 \tag{26}$$

$$\frac{\partial \Pi_B}{\partial \beta_1} = 0. \tag{27}$$

If the equilibrium is to correspond to a deformed configuration, the determinant of eqns (26) and (27) must vanish. Hence, the critical moment, $M$, may be obtained as

$$M = \frac{\pi}{L}\sqrt{\left(EI_{y1}\left[E\Gamma\left(\frac{\pi}{L}\right)^2 + GJ_1\right]\right)}. \tag{28}$$

Now, if the effects due to the web stiffeners and the batten plates in both the moment of inertia and torsional constant are neglected, the relations $I_{y1} \to I_{y0}$ and $J_1 \to J_0$ are effected and eqn (28) yields

$$M = \frac{\pi}{L}\sqrt{\left(EI_{y0}\left[E\Gamma\left(\frac{\pi}{L}\right)^2 + GJ_0\right]\right)} \tag{29}$$

and the result agrees with eqn (5.96) of Chajes (1974).

## 5. BUCKLING OF BOTH ENDS FIXED I BEAMS IN UNIFORM BENDING

A member is now considered, the ends of which are free to rotate about the horizontal axis, but fully restrained against all other displacements. Hence

$$\left.\begin{aligned} u = u' &= 0 \\ \beta = \beta' &= 0 \end{aligned}\right\} \quad \text{at } z = 0 \text{ and } L. \tag{30}$$

In common with Chajes (1974), $u$ and $\beta$ satisfying eqns (30) are approximated by

$$u = u_1\left(1-\cos\frac{2\pi z}{L}\right) \tag{31}$$

$$\beta = \beta_1\left(1-\cos\frac{2\pi z}{L}\right). \tag{32}$$

Substituting eqns (31) and (32) into eqn (10), the total potential energy, $\Pi_B$, stored in the beam during buckling may be written as

$$\Pi_B = \frac{L}{4}\left(\frac{2\pi}{L}\right)^2\left[E\left(\frac{2\pi}{L}\right)^2(u_1)^2 I_{y1} + E\Gamma\left(\frac{2\pi}{L}\right)^2(\beta_1)^2 + G(\beta_1)^2 J_1 - 2Mu_1\beta_1\right] \tag{33}$$

in which the cross sectional constants, $I_{y1}$ and $J_1$, may be expressed as

$$I_{y1} = I_{y0} + \underline{\frac{2}{L}\sum_{i=1}^{s} I_{ys}t_s\cos^2\left(\frac{2\pi z_{si}}{L}\right) + \frac{2}{L}\sum_{i=1}^{p} I_{yp}b_p\cos^2\left(\frac{2\pi z_{pi}}{L}\right)} \tag{34}$$

$$J_1 = J_0 + \underline{\frac{2}{L}\sum_{i=1}^{s} J_s t_s\sin^2\left(\frac{2\pi z_{si}}{L}\right) + \frac{2}{L}\sum_{i=1}^{p} J_p b_p\sin^2\left(\frac{2\pi z_{pi}}{L}\right)} \tag{35}$$

by using eqns (22) and (23) as approximate expressions for function $D$. The underlined terms in eqns (34) and (35) indicate the effects due to the web stiffeners and the batten plates.

The critical moment, $M$, is determined from the requirement for neutral equilibrium that the derivatives of $\Pi_B$ with respect to $u_1$ and $\beta_1$, respectively, must vanish. Hence

$$M = \frac{2\pi}{L}\sqrt{\left(EI_{y1}\left[E\Gamma\left(\frac{2\pi}{L}\right)^2 + GJ_1\right]\right)}. \tag{36}$$

Now, if the effects due to the web stiffeners and batten plates in the moments of inertia and torsional constant are neglected, relations $I_{y1} \to I_{y0}$ and $J_1 \to J_0$ are valid, and the result agrees with eqn (5.103) of Chajes (1974).

## 6. BUCKLING OF SIMPLY SUPPORTED *I* BEAMS IN CONCENTRATED LOAD

Consider the lateral buckling of the simply supported $I$ beam with the web stiffeners and batten plates subjected to a concentrated load, $P$, at the midspan, as shown in Fig. 4.

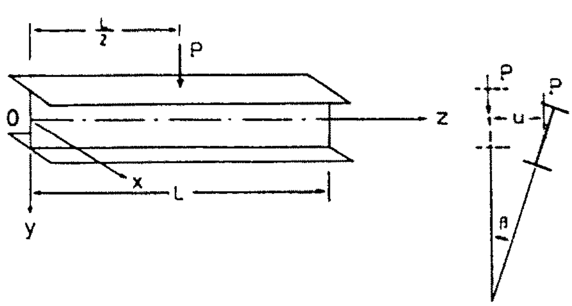


Fig. 4. Lateral buckling of *I* beam with concentrated load.

It is assumed that the concentrated load, $P$, is applied at the centroid of the cross section and that the load remains vertical as the beam buckles. The employment of this assumption simplifies the expression of the potential energy done by the external load, $P$. Fukumoto (1982) does not take this assumption into consideration when he discusses the lateral buckling of $I$ beams without web stiffeners and batten plates.

The current total potential energy, $\Pi_B$, stored in the member during buckling is obtained as

$$\Pi_B = \frac{1}{2}\int_0^L \left[EI_y(u'')^2 + E\Gamma(\beta'')^2 + GJ(\beta')^2\right]\mathrm{d}z - \int_0^{L/2} \frac{P^2 z^2 \beta^2}{2EI_y}\,\mathrm{d}z \tag{37}$$

from eqn (5.108) of Chajes (1974). It is desirable to express $u$ in terms of $\beta$ and thus reduce the number of variables in the problem. Now, if the current flexural and torsional boundary conditions of the beam are assumed to be eqns (11), then the relation

$$u'' = \frac{Pz\beta}{2EI_y} \tag{38}$$

can be applied from eqn (5.109) of Chajes (1974). Substituting eqns (1), (2) and (38) into eqn (37), the total potential energy, $\Pi_B$, yields

$$\Pi_B = \frac{1}{2}\int_0^L \left\{E\Gamma(\beta'')^2 + G\left[J_0 + \sum_{i=1}^{s} J_s D(z-z_{si}) + \sum_{i=1}^{p} J_p D(z-z_{pi})\right](\beta')^2\right\}\mathrm{d}z$$
$$- \int_0^{L/2} \frac{P^2 z^2 \beta^2}{4E\left[I_{y0} + \sum_{i=1}^{s} I_{ys} D(z-z_{si}) + \sum_{i=1}^{p} I_{yp} D(z-z_{pi})\right]}\,\mathrm{d}z. \tag{39}$$

In the above equation function $D$ appears in the denominator. This may be written as

$$\frac{1}{I_{y0} + \sum_{i=1}^{s} I_{ys} D(z-z_{si}) + \sum_{i=1}^{p} I_{yp} D(z-z_{pi})}$$
$$= \frac{1}{I_{y0}} + \sum_{i=1}^{s}{}^{*}\left[\frac{1}{I_{y0}+I_{ys}} - \frac{1}{I_{y0}}\right]D(z-z_{si})$$
$$+ \sum_{i=1}^{p}{}^{*}\left[\frac{1}{I_{y0}+I_{yp}} - \frac{1}{I_{y0}}\right]D(z-z_{pi})$$
$$+ \sum_{i=1}^{sp}{}^{*}\left[\frac{1}{I_{y0}+I_{ys}+I_{yp}} - \frac{1}{I_{y0}}\right]D(z-z_{spi}) \tag{40}$$

from the definition of the Dirac function, in which $z_{spi}$ indicates the coordinate value of $z$ at the point where the web stiffener and the batten plate are located at the same point. $\sum^{s*}$ is the sum of web stiffeners at a different location from the batten plates; $\sum^{p*}$ the sum of batten plates at a different location from the web stiffeners; and $\sum^{sp*}$ the sum of both web stiffeners and batten plates, when they are located at the same point (i.e. $z_{si} = z_{pi}$).

Equation (13) is used for the approximate function of $\beta$ satisfying the current boundary conditions. Substituting eqn (13) into eqn (39) and using eqn (40), the total potential energy, $\Pi_B$, may be written as

$$\Pi_B = \frac{L}{4}\left[E\Gamma\left(\frac{\pi}{L}\right)^4(\beta_1)^2 + GJ_1\left(\frac{\pi}{L}\right)^2(\beta_1)^2 - \frac{(\pi^2+6)P^2L^2(\beta_1)^2}{48\pi^2 EI_{y1}}\right] \tag{41}$$

in which $I_{y1}$ and $J_1$ become

$$I_{y1} = \frac{I_{y0}}{k} \tag{42}$$

$$J_1 = J_0 + \underline{\frac{2}{L}\sum_{i=1}^{s} J_s t_s \cos^2\left(\frac{\pi z_{si}}{L}\right) + \frac{2}{L}\sum_{i=1}^{p} J_p b_p \cos^2\left(\frac{\pi z_{pi}}{L}\right)} \tag{43}$$

by using the approximate relations, namely, eqns (22) and (23). Here, notation $k$ is defined as

$$\begin{aligned} k = 1 &- \underline{\frac{1}{2}\sum_{i=1}^{s}{}^{*} \frac{\alpha_s t_s}{k_0(1+\alpha_s)}\left(z_{si}\sin\frac{\pi z_{si}}{L}\right)^2} \\ &- \underline{\frac{1}{2}\sum_{i=1}^{p}{}^{*} \frac{\alpha_p b_p}{k_0(1+\alpha_p)}\left(z_{pi}\sin\frac{\pi z_{pi}}{L}\right)^2} \\ &- \underline{\frac{1}{2}\sum_{i=1}^{sp}{}^{*} \frac{\alpha_s t_s + \alpha_p b_p}{k_0(1+\alpha_s+\alpha_p)}\left(z_{spi}\sin\frac{\pi z_{spi}}{L}\right)^2} \end{aligned} \tag{44}$$

in which notations $k_0$, $\alpha_s$, and $\alpha_p$ are defined as

$$k_0 = \frac{L^3(\pi^2+6)}{48\pi^2} \tag{45}$$

$$\alpha_s = \frac{I_{ys}}{I_{y0}}$$

$$\alpha_p = \frac{I_{yp}}{I_{y0}}. \tag{46}$$

The underlined terms in eqn (44) indicate the effects due to the web stiffeners and batten plates. The coefficient 1/2 in these terms appears due to change in the integral region from the half span of the beam to the total span, such that each $\Sigma^*$ indicates the sum for the total span.

The condition that at the critical load the first variation of $\Pi_B$ must vanish gives the following critical load, $P$:

$$P = 4\left(\frac{\pi}{L}\right)^2\sqrt{\left(\frac{3EI_{y1}}{\pi^2+6}\left[E\Gamma\left(\frac{\pi}{L}\right)^2 + GJ_1\right]\right)}. \tag{47}$$

Now, if the effects due to the web stiffeners and batten plates are neglected, the relations $I_{y1} \to I_{y0}$ and $J_1 \to J_0$ are valid, and the result agrees with eqn (5.115) of Chajes (1974). The critical moment for the critical load yields

$$M = \frac{PL}{4} = 1.36\frac{\pi}{L}\sqrt{\left(EI_{y1}\left[E\Gamma\left(\frac{\pi}{L}\right)^2 + GJ_1\right]\right)}. \tag{48}$$

## 7. BUCKLING OF BOTH ENDS FIXED $I$ BEAMS IN CONCENTRATED LOAD

Consider the lateral buckling of both ends fixed $I$ beams with web stiffeners and/or batten plates, subjected to a concentrated load at the midspan. The current flexural and torsional boundary conditions are given by eqn (30).

Using the above-mentioned assumption for the concentrated load, $P$, the total potential energy, $\Pi_B$, stored in the member during buckling is given by

$$\Pi_B = \frac{1}{2}\int_0^L [EI_y(u'')^2 + E\Gamma(\beta'')^2 + GJ(\beta')^2]\,dz - \int_0^{L/2} \frac{P^2(32z^2 - 8zL)\beta^2}{64EI_y}\,dz. \tag{49}$$

In order to express $u$ by $\beta$, employing the following relation in eqn (49)

$$u'' = -\frac{M_x\beta}{EI_y} = \frac{P(4z - L)\beta}{8EI_y} \tag{50}$$

the total potential energy may be rewritten as

$$\Pi_B = \frac{1}{2}\int_0^L [E\Gamma(\beta'')^2 + GJ(\beta')^2]\,dz - \int_0^{L/2} \frac{P^2\beta^2(16z^2 - L^2)}{64EI_y}\,dz. \tag{51}$$

Now, as for the approximated function for $\beta$ satisfying the boundary equations given in eqn (30), the function given in eqn (32) is adapted. Substituting eqns (1), (2), and (32) into eqn (51) and using eqn (40), the total potential energy, $\Pi_B$, may be written as

$$\Pi_B = \frac{L}{4}\left[E\Gamma\left(\frac{2\pi}{L}\right)^4(\beta_1)^2 + GJ_1\left(\frac{2\pi}{L}\right)^2(\beta_1)^2 - \frac{P^2L^2(\beta_1)^2(\pi^2 + 34)}{64\pi^2EI_{y1}}\right] \tag{52}$$

in which $I_{y1}$ and $J_1$ are defined as

$$I_{y1} = \frac{I_{y0}}{k} \tag{53}$$

$$J_1 = J_0 + \frac{2}{L}\sum_{i=1}^{s} J_s t_s \sin^2\left(\frac{2\pi z_{si}}{L}\right) + \frac{2}{L}\sum_{i=1}^{p} J_p b_p \sin^2\left(\frac{2\pi z_{pi}}{L}\right). \tag{54}$$

Here, $k$ is defined as

$$\begin{aligned} k = 1 &- \frac{1}{2}\sum_{i=1}^{s}{}^{*} \frac{\alpha_s t_s}{k_0(1+\alpha_s)}(16z_{si}^2 - L^2)\left(1 - \cos\frac{2\pi z_{si}}{L}\right)^2 \\ &- \frac{1}{2}\sum_{i=1}^{p}{}^{*} \frac{\alpha_p b_p}{k_0(1+\alpha_p)}(16z_{pi}^2 - L^2)\left(1 - \cos\frac{2\pi z_{pi}}{L}\right)^2 \\ &- \frac{1}{2}\sum_{i=1}^{sp}{}^{*} \frac{\alpha_s t_s + \alpha_p b_p}{k_0(1+\alpha_s+\alpha_p)}(16z_{spi}^2 - L^2)\left(1 - \cos\frac{2\pi z_{spi}}{L}\right)^2 \end{aligned} \tag{55}$$

in which

$$k_0 = \frac{L^3}{4}\left(\frac{\pi^2+34}{\pi^2}\right). \tag{56}$$

The underlined terms in eqn (55) indicate the effects due to the web stiffeners and batten plates. The coefficient 1/2 in these terms also appears to be due to the change, such that each $\Sigma^*$ indicates the sum for the total span. At the critical load the first variation of $\Pi_B$ must vanish. Hence, the critical load, $P$, is

$$P = 4\left(\frac{2\pi}{L}\right)^2 \sqrt{\left(\frac{EI_{y1}}{\pi^2+34}\left[E\Gamma\left(\frac{2\pi}{L}\right)^2+GJ_1\right]\right)} \tag{57}$$

and the critical moment yields

$$M = \frac{PL}{8} = 0.474\left(\frac{2\pi}{L}\right)\sqrt{\left(EI_{y1}\left[E\Gamma\left(\frac{2\pi}{L}\right)^2+GJ_1\right]\right)}. \tag{58}$$

Now, if the effects due to the web stiffeners and batten plates are neglected, the relations $I_{y1} \to I_{y0}$ and $J_1 \to J_0$ are valid, and the result has the coefficient 0.474 for the coefficient 1.04 given by Salvadori (1956).

Then, in order to cover this difference, the following approximate function which is more accurate for $\beta$ is selected:

$$\beta = \beta_1\left(1-\cos\frac{2\pi z}{L}\right)+\beta_2\left(1-\cos\frac{4\pi z}{L}\right). \tag{59}$$

Substituting eqn (59) into eqn (51), the current total potential energy may be written as

$$\Pi_B = \frac{L}{4}\left\{E\Gamma\left[(\beta_1)^2\left(\frac{2\pi}{L}\right)^4+(\beta_2)^2\left(\frac{4\pi}{L}\right)^4\right]\right.$$

$$+G\left[J_1(\beta_1)^2\left(\frac{2\pi}{L}\right)^2+J_2(\beta_2)^2\left(\frac{4\pi}{L}\right)^2+\underline{2J_{12}\beta_1\beta_2\left(\frac{2\pi}{L}\right)\left(\frac{4\pi}{L}\right)}\right]$$

$$\left.-\frac{P^2L^2}{EI_{y0}}[\alpha_{11}k_{11}(\beta_1)^2+\alpha_{22}k_{22}(\beta_2)^2+2\alpha_{12}k_{12}\beta_1\beta_2]\right\} \tag{60}$$

in which $J_n$ ($n = 1, 2$), $J_{12}$, and $k_{nm}$ ($n = 1, 2$ and $m = 1, 2$) are defined as

$$J_n = J_0 + \underline{\frac{2}{L}\sum_{i=1}^{s} J_s t_s \sin^2\left(\frac{2n\pi z_{si}}{L}\right)+\frac{2}{L}\sum_{i=1}^{p} J_p b_p \sin^2\left(\frac{2n\pi z_{pi}}{L}\right)} \quad (\text{for } n = 1, 2) \tag{61}$$

$$J_{12} = \frac{2}{L}\sum_{i=1}^{s} J_s t_s \sin\left(\frac{2\pi z_{si}}{L}\right)\sin\left(\frac{4\pi z_{si}}{L}\right)+\frac{2}{L}\sum_{i=1}^{p} J_p b_p \sin\left(\frac{2\pi z_{pi}}{L}\right)\sin\left(\frac{4\pi z_{pi}}{L}\right) \tag{62}$$

$$k_{nm} = 1-\frac{1}{2}\frac{1}{16L^3\alpha_{nm}}\left[\underline{\sum_{i=1}^{s}{}^{*}\frac{\alpha_s t_s}{1+\alpha_s}(16z_{si}^2-L^2)\left(1-\cos\frac{2n\pi z_{si}}{L}\right)\left(1-\cos\frac{2m\pi z_{si}}{L}\right)}\right.$$

$$\underline{+\sum_{i=1}^{p}{}^{*}\frac{\alpha_p b_p}{1+\alpha_p}(16z_{pi}^2-L^2)\left(1-\cos\frac{2n\pi z_{pi}}{L}\right)\left(1-\cos\frac{2m\pi z_{pi}}{L}\right)}$$

$$\left.\underline{+\sum_{i=1}^{sp}{}^{*}\frac{\alpha_s t_s+\alpha_p b_p}{1+\alpha_s+\alpha_p}(16z_{spi}^2-L^2)\left(1-\cos\frac{2n\pi z_{spi}}{L}\right)\left(1-\cos\frac{2m\pi z_{spi}}{L}\right)}\right]$$

$$(\text{for } n = 1,2 \text{ and } m = 1,2) \tag{63}$$

and $\alpha_{11} = 0.0694519$, $\alpha_{22} = -0.00891373$ and $\alpha_{12} = -0.0226534$. For simplicity the underlined term in eqn (60) is neglected. At the critical load the first variation of $\Pi_B$ with respect to $\beta_1$ and $\beta_2$ must vanish. Hence

$$\begin{bmatrix}\dfrac{\partial\Pi_B}{\partial\beta_1}\\ \dfrac{\partial\Pi_B}{\partial\beta_2}\end{bmatrix} = \begin{bmatrix} a_1-\alpha_{11}k_{11}P_1 & -\alpha_{12}k_{12}P_1\\ -\alpha_{12}k_{12}P_1 & a_2-\alpha_{22}k_{22}P_1\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = 0 \tag{64}$$

in which notations $a_1$, $a_2$, and $P_1$ are defined as

$$a_1 = E\Gamma\left(\frac{2\pi}{L}\right)^4+GJ_1\left(\frac{2\pi}{L}\right)^2 \tag{65}$$

$$a_2 = E\Gamma\left(\frac{4\pi}{L}\right)^4+GJ_2\left(\frac{4\pi}{L}\right)^2 \tag{66}$$

$$P_1 = \frac{P^2L^2}{EI_{y0}}. \tag{67}$$

If equilibrium is to correspond to a deformed configuration, the determinant of eqn (64) must vanish. Hence

$$[\alpha_{11}k_{11}\alpha_{22}k_{22}-(\alpha_{12}k_{12})^2]P_1^2-(\alpha_{11}k_{11}+\alpha_{22}k_{22})P_1+a_1a_2 = 0. \tag{68}$$

Dividing eqn (68) by $(a_1)^2$ and using notations $k_2 = a_2/a_1$ and $P_2 = P_1/a_1$, one has

$$[\alpha_{11}k_{11}\alpha_{22}k_{22}-(\alpha_{12}k_{12})^2]P_2^2-(\alpha_{11}k_{11}+\alpha_{22}k_{22})P_2+k_2 = 0. \tag{69}$$

Solving eqn (69) for $P_2$, the critical moment $M$ yields

$$M = \frac{\sqrt{P_2}}{8}\left(\frac{2\pi}{L}\right)\sqrt{\left(EI_{y0}\left[E\Gamma\left(\frac{2\pi}{L}\right)^2+GJ_1\right]\right)}. \tag{70}$$

In order to examine the accuracy of eqn (70), show the critical moment for steel $I$ beam $H-100\times50\times5\times7$, $E = 2.1\times10^6$ kgf cm$^{-2}$, Poisson's ratio $\nu = 0.3$, and $L = 200$ cm. For the beam without web stiffeners and batten plates the torsional constant $GJ_1$ reduces to $GJ_0$, the value of $P_2$ is 62.5 kgf$^{-1}$ and the critical moment becomes

$$M = 0.99\left(\frac{2\pi}{L}\right)\sqrt{\left(EI_{y0}\left[E\Gamma\left(\frac{2\pi}{L}\right)^2+GJ_0\right]\right)}. \tag{71}$$

The result is nearly the value 1.04 given by Salvadori. The difference will be covered by using an approximate function, which is more accurate for $\beta$; however, the expression will be complicated.

## 8. NUMERICAL RESULTS

The lateral buckling of a doubly symmetric and uniform $I$ beam with web stiffeners and/or batten plates has been discussed. The obtained critical moments and critical loads have taken the same forms as the original beams without the web stiffeners and batten plates as the moment of inertia and the torsional constant are replaced with ones of original $I$ beams without web stiffeners and batten plates. Both the appropriateness of the derived critical moments and critical loads and the behavior of lateral buckling of the beam with web stiffeners or batten plates are clarified through the numerical calculations. Data used in numerical calculations take the following values: steel $I$ beam $H-100\times 50\times 5\times 7$ ($h=10$ cm, $B=5$ cm, $t_w=0.5$ cm, $t_f=0.7$ cm), Young's modulus $E=2.1\times 10^6$ kgf cm$^{-2}$ ($29.8683\times 10^6$ lb in$^{-2}$), Poisson's ratio $\nu=0.3$, $L=200$ cm (78.7402 in), $t_w=t_s=t_p=0.5$ cm (0.1969 in.), $a=b_p=2.5$ cm (0.9843 in.). The variations of the locations of the web stiffeners and the batten plates take each of five types, namely, S1–S5 for the web stiffeners and C1–C5 for the batten plates, as shown in Fig. 5. All shapes of the web stiffeners and batten plates are symmetric with respect to the $y$ axis.

The critical moments in Tables 1–5 are nondimensionalized with the critical moment of the original beam without web stiffeners and batten plates. Type 1 in Tables 1–5 shows

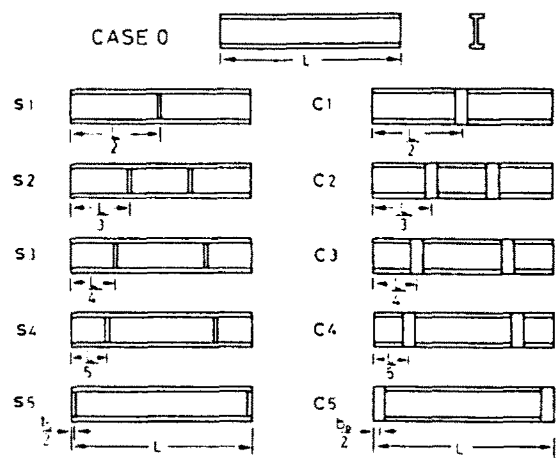


Fig. 5. Specimens in numerical calculation.

Table 1. Critical moments for a simply supported $I$ beam with web stiffeners or batten plates (uniform bending)

| Case | Critical moments Type 1 ($M/M_{01}$) | Type 2 ($M/M_{01}$) | Relative error (%) |
|---|---|---|---|
| O | 1.0 | 1.0 | 0 |
| S1 | 1.015 | 1.000 | 1.48 |
| S2 | 1.217 | 1.190 | 2.22 |
| S3 | 1.375 | 1.354 | 1.53 |
| S4 | 1.462 | 1.446 | 1.09 |
| S5 | 1.634 | 1.634 | 0 |
| C1 | 1.063 | 1.000 | 5.93 |
| C2 | 1.713 | 1.567 | 8.52 |
| C3 | 2.102 | 1.978 | 5.90 |
| C4 | 2.290 | 2.194 | 4.19 |
| C5 | 2.613 | 2.613 | 0 |

Type 1: Considers the variation of $I_y$ due to web stiffeners or batten plates.
Type 2: Neglects the variation of $I_y$ due to web stiffeners or batten plates.
Relative error = (type 1 − type 2)/type 1.

Table 2. Critical moments for both ends fixed $I$ beam with web stiffeners or batten plates (uniform bending)

| Case | Critical moments Type 1 ($M/M_{01}$) | Type 2 ($M/M_{01}$) | Relative error (%) |
|---|---|---|---|
| O | 1.0 | 1.0 | 0 |
| S1 | 1.015 | 1.000 | 1.48 |
| S2 | 1.397 | 1.387 | 0.72 |
| S3 | 1.493 | 1.493 | 0 |
| S4 | 1.458 | 1.454 | 0.27 |
| S5 | 1.030 | 1.000 | 2.91 |
| C1 | 1.063 | 1.000 | 5.93 |
| C2 | 2.120 | 2.055 | 3.07 |
| C3 | 2.302 | 2.302 | 0 |
| C4 | 2.238 | 2.211 | 1.21 |
| C5 | 1.125 | 1.003 | 10.84 |

Type 1: Considers the variation of $I_y$ due to web stiffeners or batten plates.
Type 2: Neglects the variation of $I_y$ due to web stiffeners or batten plates.
Relative error = (type 1 − type 2)/type 1.

Table 3. Critical moments for a simply supported $I$ beam with web stiffeners or batten plates (concentrated load)

| Case | Critical moments Type 1 ($M/M_{01}$) | Type 2 ($M/M_{01}$) | Relative error (%) |
|---|---|---|---|
| O | 1.0 | 1.0 | 0 |
| S1 | 1.004 | 1.000 | 0.40 |
| S2 | 1.198 | 1.190 | 0.67 |
| S3 | 1.361 | 1.354 | 0.51 |
| S4 | 1.452 | 1.446 | 0.41 |
| S5 | 1.634 | 1.634 | 0 |
| C1 | 1.020 | 1.000 | 1.96 |
| C2 | 1.621 | 1.567 | 3.33 |
| C3 | 2.029 | 1.978 | 2.51 |
| C4 | 2.236 | 2.194 | 1.88 |
| C5 | 2.613 | 2.613 | 0 |

Type 1: Considers the variation of $I_y$ due to web stiffeners or batten plates.
Type 2: Neglects the variation of $I_y$ due to web stiffeners or batten plates.
Relative error = (type 1 − type 2)/type 1.

Table 4. Critical moments for both ends fixed $I$ beam with web stiffeners or batten plates (concentrated load)

| Case | Critical moments Type 1 ($M/M_{01}$) | Type 2 ($M/M_{01}$) | Relative error (%) |
|---|---|---|---|
| O | 1.0 | 1.0 | 0 |
| S1 | 1.006 | 1.000 | 0.60 |
| S2 | 1.397 | 1.387 | 0.72 |
| S3 | 1.499 | 1.493 | 0.40 |
| S4 | 1.457 | 1.454 | 0.21 |
| S5 | 1.000 | 1.000 | 0 |
| C1 | 1.030 | 1.000 | 2.91 |
| C2 | 2.134 | 2.055 | 3.70 |
| C3 | 2.346 | 2.302 | 1.86 |
| C4 | 2.233 | 2.211 | 0.99 |
| C5 | 1.003 | 1.003 | 0 |

Type 1: Considers the variation of $I_y$ due to web stiffeners or batten plates.
Type 2: Neglects the variation of $I_y$ due to web stiffeners or batten plates.
Relative error = (type 1 − type 2)/type 1.

the critical moments considering the effects due to the web stiffeners and the batten plates on the moment of inertia about its weak axis. Conversely, type 2 shows the critical moments neglecting these effects.

Table 1 shows the critical moment of simply supported $I$ beams with web stiffeners or

Table 5. Critical moments for both ends fixed $I$ beam with web stiffeners or batten plates (concentrated load)

| Case | Type ($M/M_{01}$) | Critical moments Type ($M/M_{01}$) | Relative error (%) |
|---|---|---|---|
| O | 1.0 | 1.0 | 0 |
| S1 | 1.000 | 1.000 | 0 |
| S2 | 1.386 | 1.386 | 0 |
| S3 | 1.493 | 1.493 | 0 |
| S4 | 1.453 | 1.453 | 0 |
| S5 | 1.000 | 1.000 | 0 |
| C1 | 1.000 | 1.000 | 0 |
| C2 | 2.054 | 2.054 | 0 |
| C3 | 2.300 | 2.300 | 0 |
| C4 | 2.209 | 2.209 | 0 |
| C5 | 1.003 | 1.003 | 0 |

Type 1: Consider the variation of $I_y$ due to web stiffeners or batten plates.
Type 2: Neglects the variation of $I_y$ due to web stiffeners or batten plates.
Relative error = (type 1 − type 2)/type 1.

batten plates, subjected to uniform bending. Judging from their relative errors, the effect due to the web stiffeners on the moment of inertia about its weak axis is negligible in practice. Similarly, as the number of the batten plates is less, the effect due to the batten plates on the moment of inertia about the weak axis may also be neglected. However, the effects due to the web stiffeners and the batten plates on the torsional constant cannot be neglected. Hence, since the effects due to the web stiffeners and the batten plates on the torsional constant are remarkable, the effective location of the web stiffeners and the batten plates for the lateral buckling is governed by the value of the function $\cos^2(\pi z/L)$ in eqn (25). For the least lateral buckling mode, web stiffeners and batten plates are the most effective as their locations approach both ends, because the local increments of the torsional constant due to the web stiffeners and the batten plates are considered to constrain the rotation angle $\beta$ at both ends.

Table 2 shows the critical moments of both ends fixed $I$ beams with web stiffeners or batten plates in uniform bending. The effects due to the web stiffeners and the batten plates on the moment of inertia about its weak axis can also be neglected in practice. However, the effects due to the web stiffeners and the batten plates on the torsional constant cannot be neglected. Hence, the effective locations of the web stiffeners and the batten plates for the least lateral buckling mode are determined from the value of the function $\sin^2(2\pi z/L)$.

Tables 3–5 indicate the critical moments for both the simply supported beam and the both ends fixed beam, subjected to a concentrated load at midspan; Table 4 indicates values calculated from eqn (58) and Table 5 indicates values from eqn (71). Similarly, the effects due to the web stiffeners and the batten plates on the moment of inertia can be neglected in practice. Namely, the underlined terms in eqns (44), (55), and (63) can be neglected. Therefore, the effective locations of the web stiffeners and batten plates are determined from the values of functions $\cos^2(\pi z/L)$ and $\sin^2(2\pi z/L)$, respectively, making the torsional constant $J_1$ a maximum. On the other hand, although the values of Table 4 are slightly larger than the values of Table 5, the difference is negligible in practice. Hence, when one considers the ratio of the critical moment including the effects of web stiffeners and/or batten plate to the critical moment neglecting the effects, the difference between coefficients 0.474 and 0.99 in eqns (58) and (71), respectively, has no effect on the ratio of the critical moments. Then one may use the coefficient 1.04 given by Salvador instead of these coefficients.

The results of Tables 1–4 show that for lateral buckling the batten plates are more effective than the web stiffeners, because the existence of the batten plates changes the beam from an open cross section to a closed cross section.

Figure 6 indicates the values of functions $\cos^2(\pi z/L)$ and $\sin^2(2\pi z/L)$ determining the effective location of the web stiffeners and batten plates.

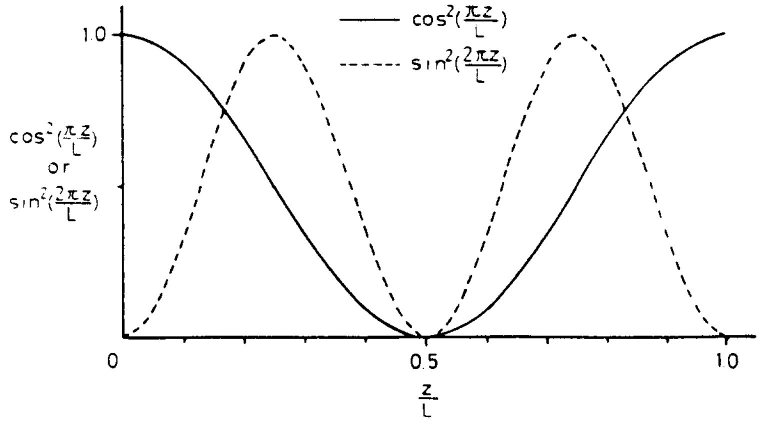


Fig. 6. Values of $\cos^2(\pi z/L)$ and $\sin^2(2\pi z/L)$.

## 9. CONCLUSIONS

The lateral buckling of a doubly symmetric and uniform $I$ beam with web stiffeners and/or batten plates has been generally presented by means of the function extending the Dirac function. The proposed critical moments have the same forms as ones of the original beam without web stiffeners and batten plates, modifying both the torsional constant and the moment of inertia about the weak axis. In practice, it has been clarified from the numerical results that the effects due to the web stiffeners and batten plates on the moment of inertia about its weak axis can be neglected, but the effects due to them on the torsional constant cannot.

Concerning the factor influencing the lateral buckling of $I$ beams, the positive factor and the negative factor are considered. The former is the web stiffeners and batten plates considered in this paper; the latter may be considered as web opening.

For beams with web opening, a similar method will be applied to the moment of inertia about its weak axis. However, since there is no variation of the torsional constant due to web opening, and since the variation of the moment of inertia about its weak axis can be neglected in practice, as stated in this paper, the expression of the lateral buckling of the beam without web opening is applicable to the beam with web opening.

Uniform beams with a solid section or closed section can be considered to be beams in which web stiffeners or batten plates, respectively, derived in this paper are continuously distributed throughout the whole of the span. Then, the well-known governing equations for the beams with these sections will be easily introduced from the theory derived here by considering that the Dirac functions $\delta(x-x_{si})$ and $\delta(x-x_{pi})$ exist continuously throughout the whole of the span of the beams. Also, for the lateral buckling of inelastic beams or of non-uniform beams, the method presented here will be applied.

The proposed result will be necessary to compare with experimental values. If a difference between the proposed result and experimental results exists, it will be based on the effective region of stiffnesses produced by web stiffeners and batten plates. In this paper this effective region has been assumed to be the thickness $t_s$ for a web stiffener or the width $b_p$ for a batten plate. If the proposed result is modified by a coefficient selected to have the good agreement with experimental values, it will be made perfect.

*Acknowledgements*—The author would like to express his appreciation to Mr Paul T. Hobelman of Kanazawa Institute of Technology for his careful reading of and effective suggestions for this manuscript. Also, the author would like to express his appreciation to Mr A. Yamamoto for his help in verifying the computer programs.

## REFERENCES

Bleich, F. (1952). *Buckling Strength of Metal Structures*. McGraw-Hill, New York.
Chajes, A. (1974). *Principle of Structural Stability Theory*. Prentice-Hall, Englewood Cliffs, New Jersey.
Frýba, L. (1972). *Vibration of Solids and Structures Under Moving Loads*. Noordhoff, Groningen.
Fukumoto, Y. (1982). *Analysis of Structural Stability* (in Japanese). Gihoodo Shuppan, Tokyo.
Galambos, T. V. (1936). *Structural Members and Frames*. Prentice-Hall, Englewood Cliffs, New Jersey.

Gjelsvik, A. (1981). *The Theory of Thin Walled Bars*. Wiley, New York.
Mikusinski, J. and Sikorski, R. (1957). The elementary theory of distributions I. *Rozpr. mat.* **12**, 1–54.
Salvadori, M. G. (1956). Lateral buckling of eccentrically loaded *I*-columns. *Trans. ASCE* **121**, 1163.
Timoshenko, S. P. and Gere, J. M. (1961). *Theory of Elastic Stability*. McGraw-Hill, New York.
Utoguch, H. (1968). *Elasticity* (in Japanese). Kyoritsu Shuppan, Tokyo.
Vlasov, V. Z. (1936). *Thin-walled Elastic Beams*, 2nd Edn. Department of Commerce, U.S.A.; Israel Program for Scientific Translations, Jerusalem.